GENERATION OF SINGULARITY CATEGORIES AND VANISHING OF COHOMOLOGY ANNIHILATORS

YUKI MIFUNE

This talk is based on joint work with Souvik Dey, Jian Liu, and Yuya Otake [4].

Let R be a commutative noetherian ring. Denote by $\operatorname{mod} R$ the category of finitely generated R-modules, and by $\mathrm{D^b}(\operatorname{mod} R)$ the bounded derived category of $\operatorname{mod} R$. The singularity category of R, introduced by Buchweitz [2], is defined as the Verdier quotient of $\mathrm{D^b}(\operatorname{mod} R)$ by the full subcategory of perfect complexes over R; that is,

$$\mathrm{D_{sg}}(R) = \mathrm{D^b}(\operatorname{mod} R)/\operatorname{thick} R.$$

This category detects the singularities of R in the sense that it is trivial if and only if R is regular.

On the other hand, the notion of (strong) generation in a triangulated category was introduced by Bondal, Rouquier, and Van den Bergh [1, 7]. Let $\mathcal{T}$ be a triangulated category. For an integer $n \geq 0$ and an object $G \in \mathcal{T}$, we define $\langle G \rangle^{\mathcal{T}}_{n+1}$ as the full subcategory of $\mathcal{T}$ consisting of objects M such that M is built out of G by taking finite direct sums, direct summands, shifts, and at most n mapping cones. We say that $\mathcal{T}$ admits a *generator* if there exists an object $G \in \mathcal{T}$ such that $\mathcal{T} = \operatorname{thick}_{\mathcal{T}} G = \bigcup_{i \geq 0} \langle G \rangle^{\mathcal{T}}_i$. Similarly, $\mathcal{T}$ is said to admit a *strong generator* if $\mathcal{T} = \langle G \rangle^{\mathcal{T}}_n$ for some $G \in \mathcal{T}$ and $n \geq 0$.

Iyengar and Takahashi [6] characterized the existence of a generator in $\mathrm{D^b}(\operatorname{mod} R)$ and $\mathrm{D_{sg}}(R)$ in terms of the openness of the regular locus of R.

Theorem (Iyengar–Takahashi [6])**.** *The following conditions are equivalent for a commutative noetherian ring R.*

(1) *The regular locus* $\operatorname{Reg}(R/\mathfrak{p})$ *contains a nonempty open subset for each* $\mathfrak{p} \in \operatorname{Spec} R$.
(2) *The regular locus* $\operatorname{Reg}(R/\mathfrak{p})$ *is open for each* $\mathfrak{p} \in \operatorname{Spec} R$.
(3) *The category* $\mathrm{D^b}(\operatorname{mod} R/\mathfrak{p})$ *admits a generator for each* $\mathfrak{p} \in \operatorname{Spec} R$.
(4) *The category* $\mathrm{D_{sg}}(R/\mathfrak{p})$ *admits a generator for each* $\mathfrak{p} \in \operatorname{Spec} R$.

The cohomology annihilator of R, denoted by $\operatorname{ca}(R)$, is defined as the set of elements $a \in R$ that annihilate $\operatorname{Ext}^n_R(M, N)$ for all $M, N \in \operatorname{mod} R$ and all sufficiently large n. This notion was introduced by Iyengar and Takahashi [5]. Dey, Lank, and Takahashi [3] characterized the existence of a strong generator in $\mathrm{D^b}(\operatorname{mod} R)$ in terms of the nonvanishing of the cohomology annihilator.

Theorem (Dey–Lank–Takahashi [3])**.** *The following conditions are equivalent for a commutative noetherian ring R.*

(1) *The category* $\mathrm{D^b}(\operatorname{mod} R/\mathfrak{p})$ *admits a strong generator for each* $\mathfrak{p} \in \operatorname{Spec} R$.
(2) *One has* $\operatorname{ca}(R/\mathfrak{p}) \neq 0$ *for each* $\mathfrak{p} \in \operatorname{Spec} R$.

In this talk, we introduce a new form of cohomology annihilator and characterize the existence of a generator in the singularity category in terms of its vanishing. This provides a new condition equivalent to those given by Iyengar and Takahashi [6]. Furthermore, by relating this cohomology annihilator to the annihilator of the singularity category, we obtain another criterion for the existence of a strong generator in the singularity category. As an application, in the case where the Krull dimension is at most one, we obtain a singularity category version of the result of Dey, Lank, and Takahashi [3] for the bounded derived category.

References

[1] A. Bondal; M. van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, *Mosc. Math. J.* **3** (2003), no. 1, 1–36, 258.

[2] R.-O. Buchweitz, Maximal Cohen–Macaulay modules and Tate cohomology, With appendices and an introduction by L. L. Avramov, B. Briggs, S. B. Iyengar and J. C. Letz, Math. Surveys Monogr. **262**, *American Mathematical Society, Providence, RI*, 2021.

[3] S. Dey; P. Lank; R. Takahashi, Strong generation for module categories, *J. Pure Appl. Algebra* **229** (2025), no. 10, Paper No. 108070, 16pp.

[4] S. Dey; J. Liu; Y. Mifune; Y. Otake, Annihilation of cohomology and (strong) generation of singularity categories, preprint (2025), arXiv:2503.24186v1.

[5] S. B. Iyengar; R. Takahashi, Annihilation of cohomology and strong generation of module categories, *Int. Math. Res. Not. IMRN* (2016), no. 2, 499–535.

[6] S. B. Iyengar; R. Takahashi, Openness of the regular locus and generators for module categories, *Acta Math. Vietnam.* **44** (2019), no. 1, 207–212.

[7] R. Rouquier, Dimensions of triangulated categories, *J. K-Theory* **1** (2008), no. 2, 193–256.

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya 464-8602, Japan

Email address: yuki.mifune.c9@math.nagoya-u.ac.jp